RESEARCH STATEMENT

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1. Introduction

Vertex operators algebras (VOAs) are remarkable structures appearing at the intersection of many exciting developments across many fields of mathematics and physics. Originally VOAs arose in the 1980s, and are generalizations of commutative algebras equipped with a derivation. Since being defined axiomatically by Borcherds [7], they have found applications to finite groups, Lie theory, partition identities, modular forms, and algebraic geometry. After providing basics of VOA terminology (2), and motivating the study of 2-parameter universal objects (3), I discuss my work on the following topics:

(1) Rectangular orthosymplectic $\mathcal{W}$-infinity algebras of types $\mathfrak{sp}_2$ and $\mathfrak{so}_2$ (3.2).
(2) $N = 2$ supersymmetric $\mathcal{W}$-infinity algebra (3.3).
(3) Minimal reduction of the orthosymplectic $\mathcal{W}$-algebras of types $\mathfrak{sp}_2$ and $\mathfrak{so}_2$, and the associated $\mathcal{W}$-infinity (4.1).

Lastly, I mention some future directions of my research (5).

2. Basics of VOAs

Vertex algebras (VAs) are a generalization of commutative and associative differential algebras. More precisely, vertex algebra V is a vector space that is $\frac{1}{2}\mathbb{Z}$-graded, endowed with a map

$$Y : a \in V \mapsto a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \in \mathrm{End}_V[[z^{\pm 1}]], \quad a(z)v \in V((z)),$$

derivative ∂ with $(\partial a)(z) = \frac{d}{dz}a(z)$, and so that any two fields $a(z)$ and $b(w)$ *quantum commute*

$$(z-w)^N[a(z), b(w)] = 0, \quad N >> 0.$$

If for all pairs a, b in $[a(z), b(w)] = 0$, then V reduces to a commutative differential ring [16]. Any $a \in V$ gives rise to $a_{(n)} \in \mathrm{End}_V$, which in turn gives rise to a bilinear maps called the n^{th}-products. Product $a_{(-1)}b$ gives rise to multiplication called the *normally ordered product*, while the nonnegative products measure the non-associativity and non-commutativity of the normally ordered product. All the structure of VA is encoded in the operator product expansion (OPE), which we write as follows

$$a(z)b(w) \sim \sum_{n=0}^{\infty} a(w)_{(n)}b(w)(z-w)^{-n-1}.$$

Many categorical and universal algebraic notions such as homomorphisms, ideals, simplicity, quotients, modules, etc., are clear. A subset $S = \{a_i\} \subset V$ is said to *generate* V if V is spanned by all nonassociative words in the letters a_i and n^{th} products. We say that S *strongly generates* V if V is can be generated by S by using only negative n^{th} products. We say that a V is of

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type $\mathcal{W}(n_1^{m_1}, n_2^{m_2}, \dots)$ if it has a strong generating set consisting of m_1 fields of weight n_1, m_2 of weight n_2 etc.

2.1. **Conformal structure.** Many vertex algebras V admit an action of the Virasoro algebra, which is a central extension of the Lie algebra of polynomial vector fields on the circle. As a Lie algebra, it is generated by operators $\{L_{(n)} = -t^n \frac{d}{dt} : n \in \mathbb{Z}\}$ and a central element κ. Often, there is an element $L \in V$ such that operators $\{L_{(n)} : n \in \mathbb{Z}\}$ satisfy commutation of the Virasoro algebra

$$[L_{(n)}, L_{(m)}] = (n-m)L_{(n+m)} + \frac{m(m^2-1)}{12}\kappa\delta_{n+m,0}.$$

We require L_0 to act diagonalizably on V, and $L_{(-1)} = \partial$. Such an L is called a *conformal* or *Virasoro vector*, and κ acts by a scalar called the *central charge*. The eigenspace decomposition of V under L_0 is called the *conformal weight gradation*, A vertex algebra equipped with a conformal structure is called a *vertex operator algebra* (VOA)[22].

2.2. **Fundamental examples.** Simplest examples of VOAs are given by *free field algebras*; these include Heisenberg algebra π, $\beta\gamma$-system, bc-system, and symplectic fermion algebra. For example, the rank n Heisenberg algebra π^n is strongly generated by fields $\{a^i\}_{i=1}^n$ with OPEs

$$a_i(z)a_j(w) \sim \delta_{i,j}(z-w)^{-2}.$$

For a simple Lie algebra $\mathfrak{g}$, the *universal affine* VOA $V^k(\mathfrak{g})$ at level k, and its simple quotient $L_k(\mathfrak{g})$, are strongly generated by $\{a^i\}_{i\in S}$, where S spans $\mathfrak{g}$, with OPEs

$$a_i(z)a_j(w) \sim k(a_i, b_j)(z-w)^{-2} + [a_i, a_j](z-w)^{-1},$$

where $(\cdot,\cdot)$ is the Killing form of $\mathfrak{g}$.

An important common generalization of affine and Virasoro VOAs are a class of VOAs known as $\mathcal{W}$*-algebras*. These are 1-parameter VOAs obtained via *quantum Drinfeld Sokolov reduction* [21]

$$\mathcal{W}^k(\mathfrak{g}, \mathbb{O}_\lambda) = H(V^k(\mathfrak{g}), \mathbb{O}_\lambda),$$

associated with Lie algebra $\mathfrak{g}$ and nilpotent orbit $\mathbb{O}_\lambda$ [13, 12, 17, 21]; nilpotent orbits $\mathbb{O}_\lambda$ are in bijection with partitions, satisfying some combinatorial conditions. For example, Virasoro algebra is obtained by $\mathcal{W}^k(\mathfrak{sl}_2, \mathbb{O}_2)$. These algebras have arisen across many areas of mathematics and physics, including integrable systems [2, 6, 5, 20, 11], the conformal field theory to higher spin gravity duality [18, **?**], the AGT correspondence [**?**], and the quantum geometric Langlands program [15, 3].

2.3. **Methods of construction.** There are also some standard ways to construct new VOAs from old ones. The *orbifold* construction begins with a VOA V and a subgroup of automorphisms G; then invariant subalgebra V^G is called G-orbifold of V. Similarly, the *coset* construction associates to a VOA V and a subalgebra A, the subalgebra $\mathrm{Com}(A, V) \subseteq V$ which commutes with A, that is

$$\mathrm{Com}(A, V) = \{v \in \mathcal{V} : \ \forall a \in A \quad [a(z), v(w)] = 0\}.$$

This is the analogue of commutant construction in the classical associative algebra setting. When A is an affine algebra $V^k(\mathfrak{g})$ we refer to this as an *affine coset*. Both $V^G \to \mathcal{V}$ and $A \otimes \mathrm{Com}(A, \mathcal{V}) \to \mathcal{V}$, are conformal embeddings, that is, $\mathcal{V}^G$ and $A \otimes \mathrm{Com}(A, \mathcal{V})$ have the same Virasoro vector as V.

3. Universal two-parameter algebras

This section is motivated by the quest to understand structure of a general $\mathcal{W}$-algebra. This is worthwhile, as it is expected that $\mathcal{W}$-algebras are basic building blocks of many VOAs. Direct approach is infeasible due to nonlinear nature of OPEs among strong generators. One promising direction is through the study of *universal objects*, which are certain VOAs with *infinite* strong generating type. This perspective offers unified view of many $\mathcal{W}$-algebras and beyond.

We motivate these with an observation: $\mathcal{W}$-algebras fall into infinite families, organized by a "common" partition pattern; these are obtained by lifting partition λ by a rectangle of *arbitrary* height, and length matching that of λ. Then the type of resulting family follows a *uniform* pattern; further, it is not hard to see that structure constants are *rational* functions in "height", and thus can be analytically continued.

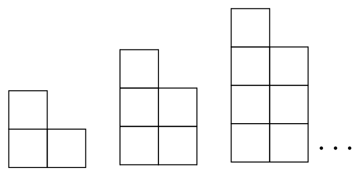

Figure 1. Partitions of increasing height $\{2,1\},\{3,2\},\{4,3\},\dots$.

One may then ask, are there some *two-parameter* algebras of this *infinite type*, which give rise to these as quotients? The answer is yes, and we refer to these as $\mathcal{W}$-infinity universal algebras. In fact, beyond $\mathcal{W}$-algebras, there are many more 1-parameter quotients; these break up by types, where in type A we have *two* $\mathbb{N}\times\mathbb{N}$-families, and in orthosymplectic types *eight* $\mathbb{N}\times\mathbb{N}$-families. Associated to each of these quotients, corresponds a *truncation curve*, which is a special algebraic relation between two parameters of the corresponding $\mathcal{W}$-infinity.

These $\mathcal{W}$-infinity type algebras have remarkable properties and have many incarnations. The most understood case is that of $\mathcal{W}_\infty$ described below; it is closely related to a number of other algebraic structures that arise in different contexts [4, 23, ?, 25]. It is expected that similar connections extend to all $\mathcal{W}$-infinity algebras.

3.1. **Single columns.** Here we consider the niloptents orbits $\mathbb{O}_\lambda$, for $\lambda=\{n,1^r\},\{n|1^r\}$. The

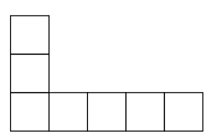

Figure 2. A Young diagram $\{3,1^4\}$, associated with $\mathbb{O}_{3,1^4}$ in $\mathfrak{sl}_7$ and $\mathbb{O}_{3|1^4}$ in $\mathfrak{sl}_{3|4}$.

associated $\mathcal{W}$-algebras are said to be of *hook-type*. This family interpolates many well-studied families, such as the principal, subregular, minimal, and affine algebras. We now describe these in greater detail in the type A, and then in orthosymplectic types.

3.1.1. *Type A.* A long-standing conjecture in physics literature [19] asserted the existence and uniqueness of a universal structure 2-parameter

$$\mathcal{W}_\infty:\mathcal{W}(2,3,4,\dots).$$

Under additional mild conditions, it was proven by Linshaw that such a VOA indeed exists [?]. The principal $\mathcal{W}$-algebras $\mathcal{W}^k(\mathfrak{sl}_n,\mathbb{O}_n)$ arise as 1-parameter quotients, however there exists many more. Consider the hook-type $\mathcal{W}$-algebras in type A,

$$\mathcal{W}^\psi(n,m):=\mathcal{W}^k(\mathfrak{sl}_{n+m},\mathbb{O}_{n,1^m}),\quad \mathcal{W}^\psi(n|m):=\mathcal{W}^k(\mathfrak{sl}_{n|m},\mathbb{O}_{n|1^m}),\quad \psi=k+{}^\vee h,$$

and their affine cosets

$$\mathcal{C}^{\psi}(n,m) := \mathrm{Com}(V^{\psi-m}(\mathfrak{gl}_m), \mathcal{W}^{\psi}(n,m)), \quad \mathcal{C}^{\psi}(n|m) := \mathrm{Com}(V^{\psi+m}(\mathfrak{gl}_m), \mathcal{W}^{\psi}(n|m)).$$

In fact, one can extend this family to the case $n=0$, defined by the coset

$$\mathcal{C}^{\psi}(0,n) := \mathrm{Com}(V^{k}(\mathfrak{gl}_n), V^{k-1}(\mathfrak{sl}_n)\otimes bc^n), \quad \mathcal{C}^{\psi}(0|n) := \mathrm{Com}(V^{k}(\mathfrak{gl}_n), V^{k+1}(\mathfrak{sl}_n)\otimes \beta\gamma^n).$$

These can be seen to have correct generating type[1], and so arise as quotients of $\mathcal{W}_\infty$. Therefore one obtains a unified perspective on many well-studied VOAs, and many more new ones.

(1) The algebras $\mathcal{C}^{\psi}(0|m)$ and $\mathcal{C}^{\psi}(0,m)$ are the well-known GKO-cosets; the celebrated namesake case $\mathcal{C}^{\psi}(0,2)$ is the Goddard-Kent-Olive coset [?].

(2) The algebras $\mathcal{C}^{\psi}(1|m)$ and $\mathcal{C}^{\psi}(1,m)$ are the type A generalized-parafermions

$$\mathcal{C}^{\psi}(1,m) := \mathrm{Com}(V^{k}(\mathfrak{gl}_m), V^{k}(\mathfrak{gl}_{1+m})).$$

These form building blocks of affine algebras $V^k(\mathfrak{gl}_n)$, i.e.

$$\bigotimes_{i=1}^{n} \mathcal{C}^{\psi}(1,i) \hookrightarrow V^k(\mathfrak{gl}_{n+1}),$$

and therefore are regarded as chiral analogues Gelfand-Tsetlin subalgebras of $U(\mathfrak{gl}_n)$ [?].

(3) The algebras $\mathcal{C}^{\psi}(2|n)$ and $\mathcal{C}^{\psi}(2,n)$ are the affine cosets of minimal $\mathcal{W}$-algebras in type A.

A priori, all 1-parameter quotients appear distinct. This is *not* the case. There exists a remarkable family of isomorphisms, known as *triality*, proven by Creutzig and Linshaw [9],

$$\mathcal{C}^{\psi}(n|m) \cong \mathcal{C}^{\psi'}(m|n) \cong \mathcal{C}^{\psi''}(n-m,m), \quad \psi\psi''=1, \quad \frac{1}{\psi}+\frac{1}{\psi'}=1. \tag{3.1}$$

(1) The isomorphism $\mathcal{C}^{\psi}(n|0) \cong \mathcal{C}^{\psi''}(n,0)$ is due to Feigin and Frenkel [14],

$$\mathcal{W}^k(\mathfrak{sl}_n, f_n) \cong \mathcal{W}^{\ell}(\mathfrak{sl}_n, f_n), \quad (k+n)(\ell+n)=1.$$

(2) The isomorphism $\mathcal{C}^{\psi}(n|0) \cong \mathcal{C}^{\psi''}(0,n)$ is the coset-realization of Arakawa, Creutzig, and Linshaw [?],

$$\mathcal{W}^{\ell}(\mathfrak{sl}_n, f_n) \cong \mathrm{Com}(V^{k+1}(\mathfrak{gl}_n), V^k(\mathfrak{sl}_n)\otimes bc^n), \qquad \ell+h^\vee = \frac{k+h^\vee}{k+h^\vee+1}. \tag{3.2}$$

Therefore we have a unified perspective on many well-known isomorphisms of VOAs, and many more new ones.

3.1.2. *Orthosymplectic types.* It was shown in [10], that there exists unique algebra VOA structure

$$\mathcal{W}^{\mathrm{ev}}_\infty : \mathcal{W}(2,4,6,\dots).$$

The analogous family of $\mathcal{W}$-algebras arising as quotients of $\mathcal{W}^{\mathrm{ev}}_\infty$ are the $\mathcal{W}^k(\mathfrak{sp}_{2n}, \mathbb{O}_{2n})$ and $\mathcal{W}^k(\mathfrak{so}_{2n+1}, \mathbb{O}_{2n+1})$. More generally, consider the orthosymplectic hook-type algebras, displayed in Table (1). Each such algebra has an affine subVOA; taking an affine coset[2] yields a 1-parameter quotient of $\mathcal{W}^{\mathrm{ev}}_\infty$. We denote these algebras by $\mathcal{C}^{\psi,\mathrm{ev}}(n,m)$ and $\mathcal{C}^{\psi,\mathrm{ev}}(n|m)$. Thus again unifying many well-known algebras.

(1) The algebras $\mathcal{C}^{\mathrm{ev},\psi}(0|n)$ and $\mathcal{C}^{\mathrm{ev},\psi}(0,n)$ are the well-known GKO-like cosets of orthosymplectic types.

[1]Modulo a mild assumptions of weak generation property.

[2]Up to a $\mathbb{Z}_2$-orbifold in the orthogonal cases.

TABLE 1. Orthosymplectic hook-type $\mathcal{W}$-(super)algebras.

$\mathcal{W}^{\mathrm{ev},\psi}(a,b)$ $\mathcal{W}^{\mathrm{ev},\psi}(a\|b)$	$a=0$	$a=2n$	$a=2n+1$
$b=2m$	$V^k(\mathfrak{sp}_{2m})\otimes bc^m$ $V^k(\mathfrak{so}_{2m})\otimes \beta\gamma^m$	$\mathcal{W}^\psi(\mathfrak{sp}_{2n+2m},\mathbb{O}_{2n,1^{2m}})$ $\mathcal{W}^\psi(\mathfrak{osp}_{2m\|2n},\mathbb{O}_{1^{2m}\|2n})$	$\mathcal{W}^\psi(\mathfrak{so}_{2n+2m+1},\mathbb{O}_{2n+1,1^{2m}})$ $\mathcal{W}^\psi(\mathfrak{osp}_{2n+1\|2m},\mathbb{O}_{2n+1\|1^{2m}})$
$b=2m+1$	$V^k(\mathfrak{so}_{2m+1})\otimes \beta\gamma^{2m+1}$ $V^k(\mathfrak{osp}_{1\|2m})\otimes \beta\gamma\otimes bc^m$	$\mathcal{W}^\psi(\mathfrak{osp}_{1\|2n+2m},\mathbb{O}_{1\|2n,1^{2m-1}})$ $\mathcal{W}^\psi(\mathfrak{osp}_{2m+1\|2n},\mathbb{O}_{1^{2m+1}\|2n})$	$\mathcal{W}^\psi(\mathfrak{so}_{2n+2m+2},\mathbb{O}_{2n+1,1^{2m+1}})$ $\mathcal{W}^\psi(\mathfrak{osp}_{2n+2\|2m},\mathbb{O}_{2n+1,1\|1^{2m}})$

(2) The algebras $\mathcal{C}^{\mathrm{ev},\psi}(1,n)$ were called *generalized parafermions of orthogonal type* in my paper [8],

$$\mathrm{Com}(V^k(\mathfrak{so}_n),V^k(\mathfrak{so}_{n+1})).$$

These are analogous to $\mathcal{C}^\psi(1,n)$, and form building blocks of affine algebras $V^k(\mathfrak{so}_n)$, in the sense that

$$\bigotimes_{i=1}^{n}\mathcal{C}^{\mathrm{ev},\psi}(1,i)\hookrightarrow V^k(\mathfrak{so}_{n+1}),$$

and are regarded as chiral analogues Gelfand-Tsetlin subalgebras of $U(\mathfrak{so}_n)$.

(3) The algebras $\mathcal{C}^\psi(2|n)$ and $\mathcal{C}^\psi(2,n)$ are the affine cosets of minimal $\mathcal{W}$-algebras in types B and C.

As in type A, not all these algebras are in distinct; this time they enjoy *four* families of isomorphisms [?], which are analogous to 3.1.

3.2. **Two-columns equal height.** Here we consider the niloptents orbits $\mathbb{O}_\lambda$, for $\lambda=\{n^2,1^r\}$. Because the partitions associated with nilpotent orbits have *two* columns, we expect these algebras are gluings of *two* $\mathcal{W}_\infty^{\mathrm{ev}}$ subalgebras; this expectation is related to reduction in stages conjecture, discussed more in (4.1). In type A, one finds one universal structure

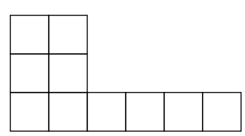

FIGURE 3. To the above $\lambda=\{3^3,1^4\}$ can corresponds to *eight* nilpotent orbits.

$$\mathcal{W}_\infty^{\mathfrak{gl}_2}:\mathcal{W}(1^4,2^4,3^4,4^4,\dots).$$

While it is a worthwhile problem to understand this algebra, our work is involved with orthosymplectic setting, where we find *two* universal structures

$$\mathcal{W}_\infty^{\mathfrak{sp}_2}:\mathcal{W}(1^3,2,3^3,4,\dots),\quad \mathcal{W}_\infty^{\mathfrak{so}_2}:\mathcal{W}(1,2^3,3,4^3,\dots).$$

Theorem 3.1. [?, ?]

(1) *$\mathcal{W}_\infty^{\mathfrak{sp}_2}$ is the unique VOA of type $\mathcal{W}(1^3,2,3^3,4,\dots)$; generators in odd weights transform as standard and in even ones as trivial $\mathfrak{sp}_2$-modules. There exists a completion of $\widehat{\mathcal{W}}_\infty^{\mathfrak{sp}_2}$, so that we have an embedding*

$$\mathcal{W}_\infty^{ev}\otimes\mathcal{W}_\infty^{ev}\hookrightarrow\widehat{\mathcal{W}}_\infty^{\mathfrak{sp}_2}.$$

(2) $\mathcal{W}_\infty^{\mathfrak{so}_2}$ *is the unique VOA of type* $\mathcal{W}(1,2^3,3,4^3,\dots)$*; two generators in even weights transform as standard, and the rest as trivial* $\mathfrak{so}_2$*-modules. There exists an an embedding*

$$\mathcal{W}_\infty^{ev}\otimes\mathcal{W}_\infty^{ev}\hookrightarrow\mathcal{W}_\infty^{\mathfrak{so}_2}.$$

The analogues of principal $\mathcal{W}$-algebras for the single-column story are the $\mathfrak{sp}_2$ and $\mathfrak{so}_2$ rectangular $\mathcal{W}$-algebras; specifically, $\mathcal{W}^k(\mathfrak{so}_{4n},\mathbb{O}_{2n,2n})$ and $\mathcal{W}^k(\mathfrak{sp}_{4n+2},\mathbb{O}_{2n+1,2n+1})$ are of $\mathfrak{sp}_2$-type, and $\mathcal{W}^k(\mathfrak{so}_{4n+2},\mathbb{O}_{2n+1,2n+1})$ and $\mathcal{W}^k(\mathfrak{sp}_{4n},\mathbb{O}_{2n,2n})$ are of $\mathfrak{so}_2$-type. More generally, we have analogues of hook-type algebras, which we call *rectangular hook-type of types* $\mathfrak{sp}_2$ *and* $\mathfrak{so}_2$, displayed in Table (2). Each such algebra contains an affine subalgebra, which commutes with $\mathfrak{sp}_2$, or $\mathfrak{so}_2$; such affine coset we denote $\mathcal{C}^{\mathfrak{sp}_2,\psi}(n,m)$, $\mathcal{C}^{\mathfrak{sp}_2,\psi}(n|m)$ in the $\mathfrak{sp}_2$-type, or $\mathcal{C}^{\mathfrak{so}_2,\psi}(n,m)$, $\mathcal{C}^{\mathfrak{so}_2,\psi}(n|m)$ in the $\mathfrak{so}_2$-type.

TABLE 2. Rectangular hook-type $\mathcal{W}$-algebras of types $\mathfrak{sp}_2$ and $\mathfrak{so}_2$.

$\mathcal{W}^{\mathfrak{sp}_2,\psi}(a,b)$ $\mathcal{W}^{\mathfrak{sp}_2,\psi}(a\vert b)$ $\mathcal{W}^{\mathfrak{so}_2,\psi}(a,b)$ $\mathcal{W}^{\mathfrak{so}_2,\psi}(a\vert b)$	$a=0$	$a=2n$	$a=2n+1$
$b=2m$	$V^k(\mathfrak{so}_{2n})\otimes\beta\gamma^{2n}$ $V^k(\mathfrak{sp}_{2n})\otimes bc^{2n}$ $V^k(\mathfrak{so}_{2n})\otimes bc^{2n}$ $V^k(\mathfrak{sp}_{2n})\otimes\beta\gamma^{2n}$	$\mathcal{W}^\psi(\mathfrak{so}_{4n+2m},\mathbb{O}_{(2n)^2,1^{2m}})$ $\mathcal{W}^\psi(\mathfrak{osp}_{4n\vert 2m},\mathbb{O}_{(2n)^2\vert 1^{2m}})$ $\mathcal{W}^\psi(\mathfrak{sp}_{4n+2m},\mathbb{O}_{(2n)^2,1^{2m}})$ $\mathcal{W}^\psi(\mathfrak{osp}_{2m\vert 4n},\mathbb{O}_{1^{2m}\vert(2n)^2})$	$\mathcal{W}^\psi(\mathfrak{sp}_{4n+2m+2},\mathbb{O}_{(2n+1)^2,1^{2m}})$ $\mathcal{W}^\psi(\mathfrak{osp}_{2m\vert 4n+2},\mathbb{O}_{1^{2m}\vert(2n+1)^2})$ $\mathcal{W}^\psi(\mathfrak{so}_{4n+2m+2},\mathbb{O}_{(2n+1)^2,1^{2m}})$ $\mathcal{W}^\psi(\mathfrak{osp}_{4n+2\vert 2m},\mathbb{O}_{(2n+1)^2\vert 1^{2m}})$
$b=2m+1$	$V^k(\mathfrak{so}_{2n+1})\otimes\beta\gamma^{2n+1}$ $V^k(\mathfrak{osp}_{1\vert 2n})\otimes\beta\gamma\otimes bc^{2n}$ $V^k(\mathfrak{so}_{2n+1})\otimes bc^{2n+1}$ $V^k(\mathfrak{osp}_{1\vert 2n})\otimes bc\otimes\beta\gamma^{2n}$	$\mathcal{W}^\psi(\mathfrak{so}_{4n+2m+1},\mathbb{O}_{(2n)^2,1^{2m+1}})$ $\mathcal{W}^\psi(\mathfrak{osp}_{4n+1\vert 2m},\mathbb{O}_{(2n)^2,1\vert 1^{2m}})$ $\mathcal{W}^\psi(\mathfrak{osp}_{1\vert 4n+2m},\mathbb{O}_{1\vert(2n)^2,1^{2m}})$ $\mathcal{W}^\psi(\mathfrak{osp}_{2m+1\vert 4n},\mathbb{O}_{1^{2m+1}\vert(2n)^2})$	$\mathcal{W}^\psi(\mathfrak{osp}_{1\vert 4n+2m+2},\mathbb{O}_{1\vert(2n+1)^2,1^{2m}})$ $\mathcal{W}^\psi(\mathfrak{osp}_{2m+1\vert 4n+2},\mathbb{O}_{1^{2m+1}\vert(2n+1)^2})$ $\mathcal{W}^\psi(\mathfrak{so}_{4n+2m+3},\mathbb{O}_{(2n+1)^2,1^{2m+1}})$ $\mathcal{W}^\psi(\mathfrak{osp}_{4n+3\vert 2m},\mathbb{O}_{(2n+1)^2,1\vert 1^{2m}})$

The following Theorems assert that $\mathcal{W}_\infty^{\mathfrak{sp}_2}$ and $\mathcal{W}_\infty^{\mathfrak{so}_2}$ unify these algebras.

Theorem 3.2. [?, ?]

(1) $\mathcal{C}^{\mathfrak{sp}_2,\psi}(n,m)$ *are pairwise non-isomorphic and arise as 1-parameter quotients of* $\mathcal{W}_\infty^{\mathfrak{sp}_2}$*. We are able to identify explicitly the commuting quotients of* $\mathcal{C}_\infty^{ev}(a,b)\otimes\mathcal{C}_\infty^{ev}(r,s)$ *in the completion of* $\hat{\mathcal{C}}^{\mathfrak{sp}_2,\psi}(n,m)$.

(2) $\mathcal{C}^{\mathfrak{so}_2,\psi}(n,m)$ *are pairwise non-isomorphic and arise as 1-parameter quotients of* $\mathcal{W}_\infty^{\mathfrak{so}_2}$*. We are able to identify explicitly the commuting quotients of* $\mathcal{C}_\infty^{ev}(a,b)\otimes\mathcal{C}_\infty^{ev}(r,s)$.

The algebras $\mathcal{C}^{\mathfrak{sp}_2,\psi}(0|m)$ and $\mathcal{C}^{\mathfrak{sp}_2,\psi}(0,m)$ are new structures which are analogues of GKO-cosets in type A and orthosymplectic types. Especially interesting is the case of $\mathcal{C}^{\mathfrak{sp}_2,\psi}(0,2m)$; this family is a new infinite family of rational VOAs containing $L_m(\mathfrak{sp}_2)$ as a subVOA, which are not isomorphic to any $\mathcal{W}$-algebra.

Theorem 3.3. [?] *Let* $\ell-1$ *be an admissible level for* $\mathfrak{sp}_{2n}$*. Then* $\mathcal{C}_\ell^{\mathfrak{sp}_2}(0,2m)$ *is strongly rational. Furthermore, it is an extension of* $\mathcal{W}_{r_1}(\mathfrak{sp}_{2k})\otimes\mathcal{W}_{r_2}(\mathfrak{sp}_{2k})$ *where*

$$r_1=-(k+1)+\frac{\ell+k+1}{2\ell+2k+1},\quad r_2=-(k+1)+\frac{\ell+k}{2\ell+2k+1}. \tag{3.3}$$

Algebras $\mathcal{C}^{\mathfrak{sp}_2,\psi}(1|m)$ and $\mathcal{C}^{\mathfrak{sp}_2,\psi}(1|m)$ are similar generalized-parafermions-like algebras

$$\mathrm{Com}(V^k(\mathfrak{sp}_{2n}),V^k(\mathfrak{sp}_{2n+2})),\quad \mathrm{Com}(V^k(\mathfrak{osp}_{1|2m}),V^k(\mathfrak{osp}_{1|2m+2}))^{\mathbb{Z}_2},$$
$$\mathrm{Com}(V^k(\mathfrak{so}_{2n}),V^k(\mathfrak{osp}_{2n|2}))^{\mathbb{Z}_2},\quad \mathrm{Com}(V^k(\mathfrak{so}_{2n+1}),V^k(\mathfrak{osp}_{2n+1|2}))^{\mathbb{Z}_2}.$$

The $\mathcal{C}^{\mathfrak{sp}_2,\psi}(1,2m)$ and $\mathcal{C}^{\mathfrak{sp}_2,\psi}(1,2m+1)$ are especially interesting because they complete the list of building blocks of orthosymplectic affine algebras; we regarded them as chiral analogues Gelfand-Tsetlin subalgebras of $U(\mathfrak{sp}_{2m})$ and $U(\mathfrak{osp}_{1|2m})$. The associated algebras of $\mathfrak{so}_2$-type are extensions of two orthogonal parafermions,

$$\mathrm{Com}(V^k(\mathfrak{so}_m), V^k(\mathfrak{so}_{m+2})) \hookleftarrow \mathcal{C}^{\mathrm{ev},\psi}(1,m) \otimes \mathcal{C}^{\mathrm{ev},\psi}(1,m-1).$$

The algebras $\mathcal{C}^{\mathfrak{sp}_2,\psi}(2,m)$ are the affine cosets of minimal $\mathcal{W}$-algebras in types D, while $\mathcal{C}^{\mathfrak{so}_2}(2,m)$ form a new minimal-like algebras for the $\mathfrak{so}_2$-family. Algebra $\mathcal{C}^{\mathfrak{sp}_2,\psi}(2,n)$ is especially interesting,

$$\mathrm{Com}(V^{k-n+4}(\mathfrak{so}_{n-4}), \mathcal{W}^k(\mathfrak{so}_n, \mathbb{O}_{2^2,1^{n-4}})).$$

Specifically, the level $k=-1$ is *non*-admissible for $V^k(\mathfrak{so}_n)$, nonetheless the simple quotient $\mathcal{W}_{-1}(\mathfrak{so}_n, \mathbb{O}_{2^2,1^{n-4}})$ is strongly rational, as we show in my work with Creutzig, Linshaw, Fasquel, and Nakatsuka.

Theorem 3.4. [?] *For $N \in \mathbb{Z}_{\geq 7}$, there exists an isomorphism of vertex algebras*

$$\mathcal{W}_{-1}(\mathfrak{so}_n, \mathbb{O}_{2^2,1^{n-4}}) \cong (L_{-1+\frac{n-4}{2}}(\mathfrak{osp}_{1|2}) \otimes \mathcal{F}^{n-4})^{\mathbb{Z}_2}.$$

In particular, algebras $\mathcal{W}_{-1}(\mathfrak{so}_{2n}, \mathbb{O}_{min})$ are strongly rational.

We remark that this proves the case of $k=-1$ of Arakawa-Moreau Conjecture [?].

3.3. **SUSY analogue.** Beyond Virasoro and affine symmetries, many $\mathcal{W}$-algebras enjoy *supersymmetry*. The $N=2$ superconformal VOA has been long well-known in physics; it is an extension of Virasoro algebra by two *odd* fields in weights $\frac{3}{2}$, and a Heisenberg field. Just as Virasoro VOA admits an extension to $\mathcal{W}_\infty$, the $N=2$ algebra extends to

$$\mathcal{W}_\infty^{N=2} : \mathcal{W}\left(1, 2^2, 3^2, 4^2, \cdots ; \left(\frac{3}{2}\right)^2, \left(\frac{5}{2}\right)^2, \left(\frac{7}{2}\right)^2, \cdots\right),$$

which we regard as a supersymmetric analogue of the $\mathcal{W}_\infty$. In my work with Creutzig, Linshaw, Song, and Sun [?] we prove the following.

Theorem 3.5. [?]

(1) *$\mathcal{W}_\infty^{N=2}$ is the unique VOA of type $\mathcal{W}(1,2^2,3^2,\cdots;(3/2)^2,(5/2)^2,\cdots)$ such that fields $\{W^{n,\perp}, W^{n,\pm}, W^{n,\top}\}$ transform as the $N=2$-diamonds for the superconformal subalgebra generated by $\{H, G^\pm, L\}$.*
(2) *It is an extension of two commuting $\mathcal{W}_\infty$ subVOAs*

$$\mathcal{W}_\infty \otimes \mathcal{W}_\infty \hookrightarrow \mathcal{W}_\infty^{N=2}.$$

The associated family of $\mathcal{W}$-algebras are the $\mathcal{W}^\psi(\mathfrak{sl}_{n+1|n}, \mathbb{O}_{n+1|n})$. The analogous hook-type families $\mathcal{W}^{N=2,\psi}(n,m)$ are displayed in Table (3). All such algebras carry an action of $\mathfrak{gl}_m$; their affine cosets are denotes $\mathcal{C}^{N=2,\psi}(n,m)$ and $\mathcal{C}^{N=2,\psi}(n|m)$. Naturally, they arise as 1-parameter quotients, however are no longer distinct.

TABLE 3. $N=2$-hook-type $\mathcal{W}$-algebras.

	$n=0$	$n>0$
$\mathcal{W}^{N=2,\psi}(n,m)$	$V^k(\mathfrak{sl}_{m+1}) \otimes bc^m$	$\mathcal{W}^k(\mathfrak{sl}_{n+1\|n+m}, \mathbb{O}_{n+1\|n,1^m})$
$\mathcal{W}^{N=2,\psi}(n\|m)$	$V^k(\mathfrak{sl}_{1\|m}) \otimes \beta\gamma^m$	$\mathcal{W}^k(\mathfrak{sl}_{n+m+1\|n+m}, \mathbb{O}_{n+1,1^m\|n})$

Theorem 3.6. [?]

(1) *$\mathcal{C}^{N=2,\psi}(n,m)$ and $\mathcal{C}^{N=2,\psi}(n|m)$ arise as 1-parameter quotients of $\mathcal{W}_\infty^{N=2}$. In particular, each one is an extension of two commuting $\mathcal{C}^\psi(a,b)$ algebras.*

(2) *We have the following isomorphisms between the 1-parameter quotients.*

$$\mathcal{C}^{N=2,\psi}(n|m) \cong \mathcal{C}^{N=2,\psi'}(m|n), \quad \mathcal{C}^{N=2,\psi}(n,m) \cong \mathcal{C}^{N=2,\psi'}(m,n), \quad \psi\psi' = 1.$$

We highlight some immediate consequences of the above Theorem. A longstanding conjecture of Ito [?, ?] says that $\mathcal{W}^k(\mathfrak{sl}_{n+1|n})$ should have a coset realization

$$\mathcal{W}^k(\mathfrak{sl}_{n+1|n}, f_{n+1|n}) \cong \mathrm{Com}(V^{\ell+1}(\mathfrak{gl}_n), V^\ell(\mathfrak{sl}_{n+1}) \otimes bc^n), \quad (k+1)(\ell+n+1) = 1.$$

which is $N = 2$ analogue of the coset realization (3.2). Our results prove it as special case of $\mathcal{C}^{N=2,\psi}(n,0) \cong \mathcal{C}^{N=2,\psi^{-1}}(0,n)$. Since the above coset is known to be rational [?], we have the following.

Theorem 3.7. [?] *For $k = -1 + \frac{1}{n+a+2}$, the simple quotient $\mathcal{W}_k(\mathfrak{sl}_{n+1|n})$ is strongly rational.*

This is a generalization of the well-known statement that the $N = 2$ minimal models, namely $\mathrm{Vir}_c^{N=2}$ for $c = \frac{3(a-1)}{a+1}$, are strongly rational [?]. We are also able to describe the module category for $\mathcal{C}_\ell(n)$, generalizing Adamovic's result on fusion rules in the module category of $N = 2$ minimal models [?].

4. Iterated reduction

Recently a conjecture has emerged, implying strong structural results for all $\mathcal{W}$-algebras; namely, that Y-algebras are the fundamental *building blocks* of all $\mathcal{W}$-algebras. Accordingly, if the associated Young diagram λ has more than one column, then the $\mathcal{W}$-algebra[3] should be an extension

$$\mathcal{W}^\psi(\mathfrak{g}, \mathbb{O}_\lambda) \hookleftarrow \bigotimes_j \mathcal{C}^{\psi_j}(\lambda_j, \sum_i \lambda_{j+i}).$$

This can be seen as a vast generalization of Gelfand-Tsetlin algebras, which is a special case $\lambda = 1^n$ or $\mathcal{W}^\psi(\mathfrak{g}, \mathbb{O}_\lambda) = V^k(\mathfrak{g})$. Our Theorems (3.2) and (3.6, (2)) support this claim in the case when λ has *two* columns. A closely related claim, which implies above, is the *iterated reduction conjecture*. To state it, choose a nilpotent element $f \in \mathbb{O}_\lambda$, which we suppose to be *decomposable* into a sum $f = f_1 + f_2$, where f_1 and f_2 are nilpotent elements, so that the completed $\mathfrak{sl}_2$-triples commute. Then the iterated reduction states that

$$\mathcal{W}(V^k(\mathfrak{g}), \mathbb{O}_{[f]}) \cong H(\mathcal{W}^k(\mathfrak{g}, \mathbb{O}_{[f_1]}), \mathbb{O}_{[f_2]}).$$

In type A, character analysis supports this claim [?]. Similarly, one expects to be able to go back using *inverse reduction*,

$$\mathcal{W}^k(\mathfrak{g}, \mathbb{O}_{[f]}) \hookrightarrow \mathcal{W}(\mathfrak{g}, \mathbb{O}_{[f_1]}) \otimes \mathcal{D}(\mathbb{O}_{[f_2]}),$$

where $\mathcal{D}(\mathbb{O}_{[f'']})$ are chiral differential operators on some affine space.

4.1. **Minimal reductions.** The simplest nontrivial case of a reduction is that of a *minimal* one. It generalizes the celebrated map

$$H_2 : V^k(\mathfrak{sl}_2) \mapsto \mathcal{W}^k(\mathfrak{sl}_2, \mathbb{O}_2) = \mathrm{Vir}^{c(k)}, \quad c(k) = 13 - 6\left((k+2)^{-1} + (k+2)\right),$$

and its inverse partner first obtained by Semikhatov [?],

$$H_{-2} : V^k(\mathfrak{sl}_2) \mapsto \mathrm{Vir}^{c(k)} \otimes \Pi,$$

[3] Or its associated completion.

$(\mathfrak{g},\mathbb{O})$	$(\mathfrak{sl}_{n+m},\mathbb{O}_{n,m})$	$(\mathfrak{so}_{2n+1},\mathbb{O}_{n^2,1})$	$(\mathfrak{sp}_{2n},\mathbb{O}_{n^2})$	$(\mathfrak{so}_{2n},\mathbb{O}_{n^2})$
$\widehat{\mathbb{O}}$	$\mathbb{O}_{n+1,m-1}$	$\mathbb{O}_{n+1,n-1}$ $\mathbb{O}_{n+2,n-2}$	$\mathbb{O}_{n+2,n-2}$ $\mathbb{O}_{n+1,n-1}$	$\mathbb{O}_{n+1,n-1}$ $\mathbb{O}_{n+2,n-2}$

TABLE 4. List of $\mathcal{W}$-algebras and associated orbits $\mathbb{O}$ and $\widehat{\mathbb{O}}$.

where Π is a localization of one $\beta\gamma$-system. In fact, theses two maps can be extended to modules, giving rise to interesting relations between representations of $V^k(\mathfrak{sl}_2)$ and Virasoro algebra at central charge $c(k)$. These processes can be performed more generally; all one needs is special element which satisfies the OPE relation $F(z)F(w) \sim 0$. Then, one can perform a minimal reduction, which is defined as the cohomology of complex

$$\mathcal{C} = \mathcal{W} \otimes bc, \quad \mathrm{d} = Q_{(0)}, \quad Q =:(b+1)F: .$$

$\mathcal{W}$-algebras $\mathcal{W}^{\mathfrak{sp}_2}(n,0)$ and $\mathcal{W}^{\mathfrak{so}_2}(n,0)$ are examples of two such families. More generally, let $\mathbb{O}$ and $\widehat{\mathbb{O}}$ be defined as in the Table (4). In our work with Fasquel and Nakatsuka [?] we show the following.

Theorem 4.1.

(1) *There is an isomorphism of vertex algerbras $H_2(\mathcal{W}^k(\mathfrak{g}),\mathbb{O}) \cong \mathcal{W}^k(\mathfrak{g},\widehat{\mathbb{O}})$. Furthermore, this map extends to a functor of $\mathcal{W}^k(\mathfrak{g},\widehat{\mathbb{O}})$-modules.*

(2) *There is an embedding of vertex algebras $\mathcal{W}^k(\mathfrak{g},\mathbb{O}) \hookrightarrow \Pi \otimes \mathcal{W}^k(\mathfrak{g},\widehat{\mathbb{O}})$.*

(3) *Minimal reduction H_2 extends to the universal object $\mathcal{W}_\infty^{\mathfrak{sp}_2}$,*

$$H_2 : \mathcal{W}_\infty^{\mathfrak{sp}_2} : \mathcal{W}(1^3, 2, 3^3, \cdots) \mapsto H_F\mathcal{W}_\infty^{\mathfrak{sp}_2} : \mathcal{W}(2^3, 3, 4^3, \cdots),$$

and the reduced universal structure is an extension of two commting even-spin subVOAs $\mathcal{W}_\infty^{ev} \otimes \mathcal{W}_\infty^{ev} \hookrightarrow H_F\mathcal{W}_\infty^{\mathfrak{sp}_2}$.

We similarly expect that analogous claim holds for $H_F\mathcal{W}_\infty^{\mathfrak{so}_2}$. We also expect that $H_F\mathcal{W}_\infty^{\mathfrak{sp}_2}$ and $H_F\mathcal{W}_\infty^{\mathfrak{so}_2}$ are themselves a universal structure.

5. Future directions

5.1. More general universal 2-parameter VOAs and associated dualities.

5.1.1. *Rectangles.* Consider the niloptents orbits $\mathbb{O}_\lambda$, for $\lambda = \{n^r, 1^m\}$. Using the stability of branching rules for $\mathfrak{g}$-modules, as rank goes to infinity, one is able to use similar methods to establish existence of all rectangular $\mathcal{W}$-infinity VOAs. In type A we find one universal structure

$$\mathcal{W}_\infty^{\mathfrak{gl}_n} : \mathcal{W}(1^{n^2}, 2^{n^2}, 3^{n^2}, 4^{n^2}, \dots).$$

It is expected to have a two $\mathbb{N}\times\mathbb{N}$ families of 1-parameter quotients, which we expect to be distinct, and be an extension n commuting copies of $\mathcal{W}_\infty$. In orthosymplectic types we find *four* rectangular $\mathcal{W}$-infinity algebras,

$$\mathcal{W}_\infty^{\mathfrak{sp}_{2n}} : \mathcal{W}(1^{S^2(2n)}, 2^{\wedge^2(2n)}, 3^{S^2(2n)}, \dots), \quad \mathcal{W}_\infty^{\mathfrak{so}_{2n}} : \mathcal{W}(1^{\wedge^2(2n)}, 2^{S^2(2n)}, 3^{\wedge^2(2n)}, \dots),$$
$$\mathcal{W}_\infty^{\mathfrak{osp}_{1|2n}} : \mathcal{W}(1^{S^2(1|2n)}, 2^{\wedge^2(1|2n)}, 3^{S^2(2n)}, \dots), \quad \mathcal{W}_\infty^{\mathfrak{so}_{2n+1}} : \mathcal{W}(1^{\wedge^2(2n+1)}, 2^{S^2(2n+1)}, 3^{\wedge^2(2n+1)}, \dots),$$

where $S^2(n)$ and $\wedge^2(n)$ denote the symmetric and exterior powers of standard representations of respective Lie algebras. We expected to find eight $\mathbb{N}\times\mathbb{N}$ families of 1-parameter quotients, all distinct. We also expect to find new rational vertex algebras arising as GKO-like cosets, with affine subVOA having a positive integer level.

5.1.2. *More SUSY analogues.* Recall that $\mathcal{W}_\infty^{N=2}$ can be seen as a SUSY analogue of $\mathcal{W}_\infty$. Similarly, algebra $\mathcal{W}_\infty^{N=1}$ algebra can be thought of as a SUSY analogue of the $\mathcal{W}_\infty^{\mathrm{ev}}$,

$$\mathcal{W}_\infty^{N=1} : \mathcal{W}\left(2^2, 4^2, 6^2, \cdots; \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \cdots\right),$$

and we expect it is an extension of two commuting copies of $\mathcal{W}_\infty^{\mathrm{ev}}$. Further, we expect it have *four* different dualities of Feigin-Frenkel type.

The supersymmetric analogue of $\mathcal{W}_\infty^{\mathfrak{sp}_2}$ is the $N = 4$ $\mathcal{W}$-infinity algebra

$$\mathcal{W}_\infty^{N=4} : \mathcal{W}\left(1^3, 2^6, 3^4, 5^6, 6^4, \cdots; \left(\frac{1}{2}\right)^3, \left(\frac{3}{2}\right)^4, \left(\frac{5}{2}\right)^4, \left(\frac{7}{2}\right)^4, \cdots\right),$$

and we expect it is an extension of *two* commuting copies of $\mathcal{W}_\infty^{\mathfrak{sp}_2}$, and to in some appropriate completion to find *four* commuting copies of $\mathcal{W}_\infty^{\mathrm{ev}}$. This algebra is connected with an exceptional simple Lie superalgebra $D(2,1;\alpha)$, just as $\mathcal{W}_\infty^{N=2}$ with $N = 2$ algebra.

5.2. **Beyond universal 2-parameter algebras.** More ambitious goal is to study 3-parameters universal objects, which are yet to be constructed. By examining their *truncation surfaces* we would find non-trivial isomorphisms of 2-parameter VOAs. By intersecting these truncation surfaces, we would find non-trivial isomorphism of 1-parameter VOAs. We believe the triality phenomenon to be general, so we expect it to extend to other universal objects over arbitrary rings. Our work on $\mathcal{W}_\infty^{N=2}$ confirms, just as the trialities of Creutzig and Linshaw, that we are at a beginning of a richer story.

One source of examples involves the diagonal cosets

$$C^{k_1,k_2}(\mathfrak{sl}_n) = \mathrm{Com}(V^{k_1+k_2}(\mathfrak{sl}_n), V^{k_1}(\mathfrak{sl}_n) \otimes V^{k_2}(\mathfrak{sl}_n)).$$

Analytically continuing parameter n, we obtain a 3-parameter VOA $C^{k_1,k_2,\nu}$. Thanks to Procesi's first and second fundamental theorems of invariant theory for the adjoint representation of $\mathfrak{sl}_n$ [24], $C^{k_1,k_2,\nu}$ can be understood as a limit of invariant rings $C^{k_1,k_2}(\mathfrak{sl}_n)$. This means that $C^{k_1,k_2,\nu}$ should have graded character

$$\frac{1}{1-q}\frac{1}{(1-q^2)^2}\frac{1}{(1-q^3)^3}\frac{1}{(1-q^4)^5}\frac{1}{(1-q^5)^7}\frac{1}{(1-q^8)^{13}}\frac{1}{(1-q^9)^{19}}\frac{1}{(1-q^{10})^{35}}\frac{1}{(1-q^{11})^{59}}\cdots$$

where exponents counts *cyclic compositions* [1].

5.3. **Supersymmetric AGT correspondence.** In [**?**] Alday, Gaiotto, and Tachikawa observed a remarkable coincidence between $N = 2$ supersymmetric guage theory in dimension 4, and two dimensional conformal field theory. Mathematically, it implies existence of a representation of a principal $\mathcal{W}$-algebra on the space of moduli space on the equivariant intersection cohomology of the moduli space of G^L spiked instantons on $\mathbb{R}^4$. In [**?**] this has been shown by Schiffman and Vasserot in the case of principal $\mathcal{W}$-algebras of type A. The relevant statement is the identification of Gaiotto state in the cohomolgy ring with the Whittaker vector in the associated module of the principal $\mathcal{W}$-algebra. It is expected that the associated $\mathcal{W}_\infty$ algebra is to act on these spaces; then it would follow that any affine coset (3.2) would act too. The key difficulty is bridging the link of $\mathcal{W}_\infty$ with an affine Yangian $Y(\widehat{\mathfrak{gl}}_1)$. In general, we expect there to be a bijection with $\mathcal{W}$-infinity structures and truncated shifted affine Yangians of appropriate affine Lie algebras. In a joint work with Creutzig we are attempting confirm this connection in special case of $\mathcal{W}_\infty^{N=2}$, thereby proving an supersymmetric generalization of the AGT correspondence.

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